Fermionic spinon and holon statistics in the pyrochlore quantum spin liquid

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The one-band Hubbard model on the pyrochlore lattice contains an extended quantum spin-liquid phase formed from the manifold of singlet dimer coverings. We demonstrate that the massive and deconfined spinon excitations of this system have fermionic statistics. Holonic quasiparticles introduced by doping are also fermions and we explain this counterintuitive but general result.

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I. INTRODUCTION

Quantum spin liquids (QSLs) have become a focus of intense activity in theory, numerics, experiment, and materials growth. Theoretical interest is driven by the possibility of understanding unconventional gapped and gapless quantum ground states, including their entanglement, topological properties, and fractional elementary excitations [1]. Although many QSL properties have been studied by considering somewhat abstract quantum dimer models (QDMs) [2, 3], Kitaev models [4], SU(N) [5], and other models, only recently have they been proven exactly in a physically relevant Hamiltonian, the one-band Hubbard model on the pyrochlore lattice [6]. In common with more abstract QSLs, the pyrochlore QSL is based on a highly degenerate ground manifold (of nearest-neighbor dimer coverings), occurring at an exactly solvable Klein point in a frustrated spin model [7, 8]. An exact treatment of perturbations about this point reveals an extended region of parameter space where the ground state is a three-dimensional (3D) QSL with massive and deconfined spinon excitations.

A fundamental question about any QSL concerns the statistics of its quasiparticles. The connection of the intrinsic spin to the statistical nature of a particle dates back to Pauli [9]. In the absence of Lorentz invariance, as in a solid, and in the presence of strong interactions, new options exist for the statistics of "emergent" low-energy quasiparticles. The best-known examples are the quasiparticles of the fractional quantum Hall effect [10–12], which have fractional (or anyonic) statistics; similar effects have been sought in high-temperature superconductors [13, 14] and other models [15], including (chiral) QSLs [16–18] and quantum critical systems [19, 20]. Although not necessarily fractional, quasiparticles in these models may nevertheless contradict the spin-statistics theorem, such as the bosonic $S = 1/2$ spinons discussed in Refs. [21, 22].

Here we investigate the quasiparticle statistics of the pyrochlore QSL. Because all states of the ground manifold are known exactly, as are all transition matrix elements, this system may be understood completely and used to extend existing QSL knowledge. We compute the statistics of spinons, demonstrating that they are fermionic. We then find that holons, the charged quasiparticles obtained by doping the QSL, are also fermions. We demonstrate that this result has a simple electronic explanation and establish the connection of these emerging fermions with gauge fields, represented by strings, as anticipated in Ref. [23].

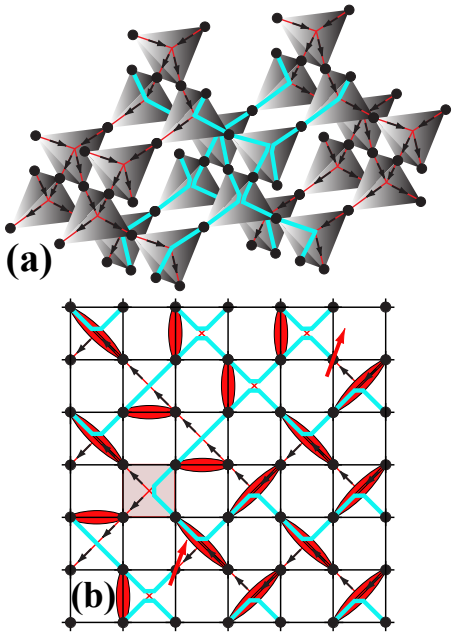

FIG. 1. (color online) (a) Pyrochlore lattice. The dimer covering (shown in Fig. 1 of Ref. [6]) is replaced both by its six-vertex representation (black arrows) and by the line representaion (light blue). Lines are drawn on all up-pointing arrows and do not cross in the (xz) plane. (b) Spinon motion represented on the checkerboard lattice, where lines are drawn on all right-pointing arrows and do not cross in $\hat{y}$ [8]. Local (spinon-dimer) processes allow the spinons to move off the line changing direction at the DT (shaded plaquette).

Although we focus for exactness on the pyrochlore QSL, our considerations regarding the fermionic statistics of both spinons and holons are ubiquitous. We will show that they are related directly to the fundamental underlying statistics of the constituent electrons. Thus our results are universal for degenerate dimer-based elec-

tronic systems, including resonating valence-bond (RVB) states, and are by no means specific only to one model.

The structure of this article is as follows. In Sec. II we review the nature and known properties of the pyrochlore QSL. In Sec. III we discuss how best to calculate the statistics of spinon excitations in the pyrochlore geometry and use this method to demonstrate that spinons are fermions. In Sec. IV we introduce the doped pyrochlore QSL and show that the resulting holon quasiparticles are also fermions; we explain this inobvious result and demonstrate its generality. In Sec. V we implement a lattice gauge-theory representation of the local conservation law of dimer number to extend our analysis of emerging fermions by including their U(1) gauge content and its representation as strings. In Sec. VI we discuss the physical relevance of strings by comparison with the line representation and clarify the issue of the local nature of quasiparticles. Section VII contains a brief summary and conclusion.

II. PYROCHLORE QUANTUM SPIN LIQUID

The pyrochlore lattice is a 3D array of corner-sharing tetrahedra [Fig. 1(a)]. The low coordination number, prevalence of triangles, and relevance of the "ice rules" [24] all contribute a wealth of phenomena in frustrated magnetism, including those of semiclassical "spin-ice" materials [25]. The existence and properties of the pyrochlore QSL arise largely from the four-site symmetry of the tetrahedra and the zero-divergence condition [two in- and two out-pointing arrows in the six-vertex representation of Fig. 1(a)] encoded in the ice rules.

Here we outline the sequence of logic proving the existence of the pyrochlore QSL [6]. 1) A physically realistic Hamiltonian, the Hubbard model at half filling with pyrochlore geometry, gives a spin model with very strong fourth-order contributions. 2) These place it close to a very high-symmetry point, a Klein point, where the Hamiltonian is intra-tetrahedal only and can be expressed as a sum of projectors. 3) All states with one dimer per tetrahedron are exact ground states of this model. 4) There is a one-to-one mapping from this manifold of states to the six-vertex model (which encodes the ice rules). 5) This ground manifold has extensive degeneracy, among whose consequences is that all states of the manifold are connected to other states by zero-dimensional (small-loop) processes. 6) The zero-divergence condition of the ice rules is a local conservation law, which results in the U(1) gauge nature of the system (Coulomb phase).

7) By constructing the submanifold maximizing the number of local dimer fluctuation processes around hexagons (i.e. involving six tetrahedra), which control the behavior under physical perturbations away from the Klein point, one may demonstrate the persistence of a highly degenerate ground manifold of $\mathcal{O}(2^{L/3})$ basis states, where L is the linear dimension of the system. Loop calculations for the physical processes linking states of the new ground manifold show that all such states gain energy from mutual resonance and that their linear combinations span all dimensions and break no lattice symmetries. 8) To complete the proof of a QSL ground state, one demonstrates that the states of the new ground manifold satisfy rigorous topological criteria, based on the presence of both local (0D) and emergent planar (2D) gauge-type symmetries, respectively of U(1) and Z_2 type. Thus one may conclude the existence of a true, zero-temperature, 3D QSL occurring over an extended region of the model parameter space around the Klein point.

Next we summarize the energetic properties of the spin excitations of the pyrochlore QSL, which arise as a necessary consequence of its nature. a) The destruction of one singlet creates a defect tetrahedron (DT) with no dimer. b) This process creates two individual spins in a net triplet state and comes at an energetic cost [6], meaning that the spin excitations are massive. c) Once created, the two individual spins may propagate separately at no further energetic cost by the rearrangement of dimers; hence it is appropriate to regard these $S = 1/2$ objects as spinons, which further are deconfined, their quantum dynamics allowing free propagation at $T = 0$. For a graphical representation of spinon motion, Fig. 1(a) illustrates both the six-vertex and line representations [8] of a dimer configuration in the ground manifold of the pyrochlore QSL [6]. For ease of visualization, we switch in Fig. 1(b) to the checkerboard (2D pyrochlore) lattice to illustrate the dimers, spinons, DTs, and their associated line representation [26].

d) Off the Klein point, free spinon propagation is constrained by "dimensional reduction:" the fact that the dimension of the ground manifold scales exponentially with L (and not L^3) makes the system effectively 2D and the spinons move normal to its fluctuating planar degrees of freedom. This situation is encapsulated by the line representation of Fig. 1(a), where the planes are horizontal and the non-Klein-point ground states have on average of one line per tetrahedron (which is the maximally degenerate submanifold of the Klein-point states). e) In addition to planar fluctuations, local processes allow free movement of spinons from one line to another, as represented on the checkerboard lattice in Fig. 1(b). As a result, their motion is fully 3D and the situation is quite different from the strict linear spinon motion of Ref. [7]. f) The connection of the U(1) gauge field to the quasiparticles of the pyrochlore QSL has not yet been explored and is one subject of the current study; in fact we will demonstrate that it is the quasiparticles which make the U(1) field manifest.

A further property of the pyrochlore QSL not discussed in Ref. [6] is the following. Gapped spinons, meaning with a singlet-triplet gap, mediate short-ranged spin-spin correlation functions. However, the ground manifold, both at and off the Klein point, is a highly degenerate set of singlet states. In this singlet manifold, dimer-dimer

correlations are algebraic, i.e. long-ranged, as shown explicitly in Ref. [8]. This type of state was classified in Ref. [27] as a "Type-II Gapped" QSL and it arises in the pyrochlore Hubbard model as a consequence of the rigorous dimerization of all spin degrees of freedom (perfect singlet formation). Any system in which this process is only approximate could not display the two contrasting paradigms for QSL nature simultaneously. However, the definition of the pyrochlore QSL as "gapped" should be regarded as semantic only, or at best probe-dependent, because it may equally be classified as an algebraic dimer liquid.

III. SPINON STATISTICS CALCULATIONS

The statistical nature of quasiparticle excitations is not only an important characteristic of any strongly correlated quantum system but a fundamental question intrinsic to our understanding of the fabric of physics, namely the defining properties of bosons, fermions, and anything in between. From the standpoint of strongly interacting systems, studies of cuprate-inspired models [28–30] and doped QDMs [31, 32] reveal that statistics are optional in 2D because flux attachment can interconvert bosons and fermions. In QDMs and also our checkerboard QSL, these are the only options, i.e. anyons are excluded.

Quasiparticle statistics are expected to be robust in 3D, but more challenging to compute because there is no unique exchange path. A definition of statistics based on the quasiparticle hopping algebra [23] was recently specialized [33] to compute the statistics of monomers in a 3D QDM from the expression $\theta_{ex} = \theta_s + \phi$. Here θ_{ex} is the flux (effective statistical angle) for exchanging two quasiparticles and ϕ, the flux due to changes required of the dimer background to restore the initial state, is computed by taking a single quasiparticle around exactly the same path. θ_s is the "true" statistical angle intrinsic to the quasiparticles.

We apply this procedure [33] to compute the statistics of $S = 1/2$ spinons moving in the SU(2) dimer background of the pyrochlore QSL. The available local moves are obtained by applying the spin Hamiltonian

$$H_t = \sum_l [\tfrac{1}{2} J_1 \mathbf{S}^2_{l,\mathrm{tot}} + \tfrac{1}{4} J_2 \mathbf{S}^4_{l,\mathrm{tot}}], \qquad (1)$$

which acts on each individual tetrahedron, l, whose total (four-site) spin is $\mathbf{S}_{l,\mathrm{tot}}$ [6]. We will show below that, if a spinon is present on tetrahedron l, the effect of H_t is to exchange the spinon position with one end of the dimer on the tetrahedron. The primary difficulty in the pyrochlore geometry is finding valid paths, composed only of these moves, which both exchange particles and restore the initial state. This requires additional moves only of the dimer background [33], which can be composed of the the minimal 8-(12-)bond loops in 2D (3D) systems, knwon as Rokhsar-Kivelson (RK) processes [2, 6].

One- and two-spinon processes valid for the considerations of exchange statistics are possible only when the quasiparticles are located next to, but not on, a dimer loop, as represented in 2D in the 1st and 7th panels of Fig. 2. Here, two-spinon exchange (left panels) alters the dimer background by one regularly-shaped RK loop and a single RK process is required to restore the initial state. When a single spinon moves on exactly the same

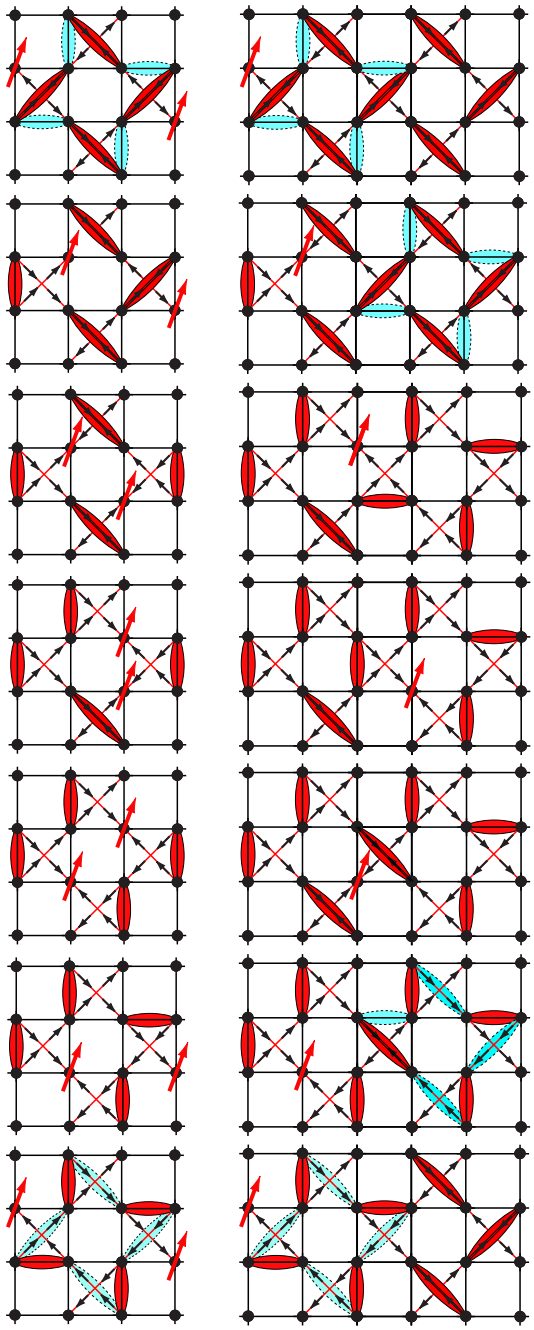

FIG. 2. (color online) Representation of spinon exchange processes on the checkerboard lattice. Left: the six steps required to exchange two spinons. Right: steps required for a single spinon to trace the same path. Red (gray) ellipses denote dimers and turquoise (light gray) ellipses the flipped dimer configuration resulting from an RK loop process.

loop (right panels), the initial dimer configuration blocks the site from which the second spinon began and the process requires a symmetrical RK loop (Fig. 2, 2nd right panel) acting in the neighboring square of the system. Restoring the initial state requires both an asymmetric RK process (6th panel) and the symmetric RK loop of the spinon path (1st and 7th panels).

We compute the associated signs from a sign convention [Fig. 3(a)] with arrows oriented upwards ($+\hat{z}$) on a bond as the primary criterion and along $+\hat{x}$ as secondary. Loops are traversed clockwise. The sign contributed by each local process may be deduced only from the triangle of sites formed by the moving spinon and dimer. The relevant process in H_t, $\vec{S}_i.\vec{S}_k$, cast as the permutation operator $P_{ik} = 2(\vec{S}_i.\vec{S}_k + 1/4)$, acts on the three-spin state $|d_{ij}\rangle|\uparrow_k\rangle \equiv \frac{1}{\sqrt{2}}(|\uparrow\downarrow\rangle - |\downarrow\uparrow\rangle)|\uparrow\rangle$ to produce the permuted state $\frac{1}{\sqrt{2}}(|\uparrow\downarrow\uparrow\rangle - |\uparrow\uparrow\downarrow\rangle) \equiv |\uparrow_i\rangle|d_{kj}\rangle$. This process changes the effective singlet sign on the loop, which traverses the sites in the order ijk; the same applies for a spinon of opposite spin, $|\downarrow_k\rangle$. The overall statistical angles are the products of these sign exchanges along the even-length loops of Fig. 2 with the factors due to the RK loops.

Because the symmetric RK loop contributes $+1$, the overall factor for two-particle exchange is $e^{i\theta_{ex}} = (-1)^6 \times (1)$, whence $\theta_{ex} = 0$. For the single-spinon loop, the six local spinon-dimer exchange moves and the accompanying symmetric RK process contribute the same factor. However, the two additional RK processes of necessity have different shapes, which is critical in that the asymmetrical RK loop contributes a factor of -1. Thus we conclude that $\phi = \pi$ and hence $\theta_s = \pi$, meaning spinons are fermions.

This result is due only to the rearrangement of dimers, i.e. emerging fermionic statistics [33, 34] are purely a consequence of the dimer background. Results for the 3D pyrochlore [Fig. 3(b)] are identical to those in 2D [Fig. 2]. The processes involve even numbers of local spinon moves and hexagonal RK loops symmetrical other than a single asymmetrical loop in the one-spinon process.

To understand the origin of this result we consider an operator representation. Dimer singlets are bosonic objects created by an operator $d^\dagger_{ij} = \frac{1}{\sqrt{2}}(c^\dagger_{i\uparrow}c^\dagger_{j\downarrow} - c^\dagger_{i\downarrow}c^\dagger_{j\uparrow})$, which is manifestly a composite of two fermionic electrons. At half-filling, there is no net charge motion and for processes within the ground manifold there is a meaningful sense in which $d^\dagger_{ij} = \frac{1}{\sqrt{2}}(f^\dagger_{i\uparrow}f^\dagger_{j\downarrow} - f^\dagger_{i\downarrow}f^\dagger_{j\uparrow})$, where $f^\dagger_{i\sigma}$ creates a spinon, an entity with only spin degrees of freedom. These spinon operators describe completely the changing spin correlations between sites during all processes changing the dimer coverings. As a consequence of perfect dimer singlet formation, spinons are fully spin-polarized, i.e. they have zero entanglement with any dimer spins.

However, these same dimer rearrangement processes, specified above by the permutation operator, P_{ij}, are brought about by the real exchange of electrons, meaning that not only the spin but also the orbital degrees of freedom are exchanged. Because this process exchanges electrons described by $c^\dagger_{i\sigma}$, it is not surprising that the spinons are fermionic. To state this result more explicitly, the many-body electronic wave function, $|\psi\rangle$, of the system changes sign under simultaneous exchange of all the orbital and spin degrees of freedom of any two electrons ($i \leftrightarrow j$, $\sigma_i \leftrightarrow \sigma_j$). Because the spinons correspond to two unentangled electrons of the same spin polarizations then by the fermionic character of the electrons, the exchange of spinons at sites i and j leads to a minus sign. Beyond the effective spinon description, and certainly at the scale of the Hubbard repulsion, U, appearing in the denominators of J_1 and J_2 (1), the preeminence of electronic statistics is clear; fermionic spinons are "emergent" quasiparticles in the sense of the low-energy limit.

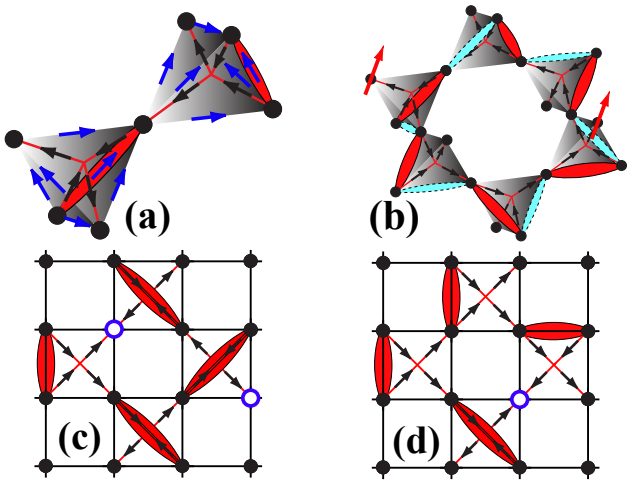

FIG. 3. (color online) Representation of (a) the sign structure adopted for the 3D pyrochlore; (b) two spinons beside a flippable loop allowing an exchange path around a single pyrochlore hexagon; (c) two holons on a flippable loop allowing an exchange path around a single checkerboard square; (d) a single holon on the same path.

IV. HOLE DOPING AND HOLON STATISTICS

Next we consider the statistics of hole-like quasiparticles. Charge degrees of freedom arise on doping into the half-filled band and incur an energy penalty from the concomitant introduction of DTs. Spinon deconfinement causes immediate spin-charge separation [6], but "holons" also propagate in the dimer background through the kinetic term $-t\sum_{\langle ij\rangle,\sigma} c^\dagger_{i\sigma}c_{j\sigma}$ of the Hubbard model. An important point not made in Ref. [6] is that the holon band width remains of order t because, unlike the familiar square-lattice case, no local magnetic ordering acts to suppress hole motion.

Proceeding as for spinons, the action of $\sum_\sigma c^\dagger_{k\sigma}c_{i\sigma}$ on state $|d_{ij}\rangle|0_k\rangle = \frac{1}{\sqrt{2}}(|\uparrow\downarrow 0\rangle - |\downarrow\uparrow 0\rangle)$ yields $\frac{1}{\sqrt{2}}(|0\downarrow\uparrow\rangle - |0\uparrow\downarrow\rangle) \equiv |0_i\rangle|d_{kj}\rangle$, i.e. there is an exact permutation

of the hole state $|0_k\rangle$ with one end of the dimer, leading again to a sign-change along the loop. The full situation, depicted in Figs. 3(c) and 3(d) for two representative steps of the exchange and one-holon processes, is identical to that for spinons. The local quasiparticle moves give even numbers of -1 factors and the RK loops determine the statistics. We conclude that holons are fermions.

The most transparent way to understand holon statistics is to introduce them as a pair; we defer a discussion of the complexities in this process to Sec. VIB. This pair replaces a single dimer, the operator expression of the process being $h_i^\dagger h_j^\dagger d_{ij}$, with $h_i^\dagger = \sum_\sigma c_{i\sigma}$. Like spinons, both holons are entirely decoupled from the remaining dimers and hence their motion under all local dimer processes gives them fermionic statistics. A holon is simply the absence of an electron, is therefore fermionic, and may more accurately be termed a "hole."

In more detail, the hole at site i may be regarded as an electron that was initially in a localized orbital state at i but subsequently ejected from the system, retaining in the process its original spin degrees of freedom and creating a state charged relative to its background. When two such excited electrons, associated with initially localized states at sites i and j, are ejected from the many-body wave function, $|\psi\rangle$, exchanging their orbital (and charge) degrees of freedom leads to $h_i^\dagger h_j^\dagger P_{ij}|\psi\rangle = -h_i^\dagger h_j^\dagger|\psi\rangle$, i.e. hole permutation is fermionic. We observe in addition that the action of $h_i^\dagger$ on the dimer state $|d_{ij}\rangle$ leaves $\frac{1}{\sqrt{2}}(c_{j\downarrow}^\dagger - c_{j\uparrow}^\dagger)$, which specifies a spinon state, of no relative charge and perfect spin polarization along $-\hat{x}$. We stress that there is no sense in which spinons and holons are fractionalized electrons; instead they are fractionalized dimers, these being the fundamental objects of the undoped ground state.

The key to this result is that both spinon and holon quasiparticles are fully specified by electron operators. Thus the underlying reason for their fermionic statistics is explicit. This is not a consequence of any special model such as the pyrochlore QSL, which we use here to make our statements rigorous. Quite generally, in any electronic dimer state, however complex, both holon and spinon dynamics arise ultimately from fermionic electron motion. This result is completely universal, and it is clear from the definition of the quasiparticles in terms of electronic operators that it overrides all details such as longer-ranged dimers and local processes, overcomplete basis sets, or excitations ("visons") connecting different topological sectors.

More subtle is the question of whether quasiparticle statistics can be dictated by the signs of the loops, meaning the overlap matrix elements between different dimer configurations in the ground manifold. In Ref. [31] it was shown that the statistics of holons doped in 2D QDMs can be exchanged from fermionic to bosonic by exchanging the sign of the quantum dimer kinetic term, which is a matrix element. This result, equivalent to the discussion of flux attachment [29, 30], was later generalized to include the signs of all loop processes in the system [32]. Although the sign of the interaction term is fixed in our electronic model, it is possible to change the phase relation between the different dimer configurations appearing in the linear superposition of coverings, $|\psi\rangle = \sum_a c_a|\psi_a\rangle$, making up the ground state [6]. However, unlike the situation in 2D, where even in frustrated models it may be possible to exchange the relative signs of all pairs of dimer coverings, $|\psi_a\rangle$, in the ground state differing by short loops [8], it is manifestly impossible to achieve this for all the highly overlapping short loops arising in the 3D system [6]. Thus we must conclude that quasiparticle statistics are not arbitrary in 3D; if the electronic state has a valid dimer description, then the quasiparticles have explicit representations in terms of electron operators and they have fermionic statistics. The only possible exceptions arise in 2D systems allowing the attachment of statistical flux, which has an alternative interpretation in terms of a loop sign ambiguity.

V. FERMIONS, STRINGS, AND GAUGE FIELDS

Emergent fermionic quasiparticles, their gauge symmetry, and the extended entities ("strings") they form are the three fundamental concepts introduced in Ref. [23]. The general case of arbitrary quasiparticle number reflects the gauge symmetries, or local conservation laws, of the system. The only strict local conservation law, the zero-divergence condition, is one dimer per tetrahedron. In the presence of quasiparticles and DTs, one has $n_{di} + n_{DTi} = 1$ for the dimer and DT numbers on every tetrahedron i. Each DT introduces two free quasiparticles, restricted only by a global conservation law, $\Sigma_i[(n_{si} + n_{hi}) - 2n_{DTi}] = n_s + n_h - 2n_{DT} = 0$. This expression corresponds directly to the "effective charges" of spinons (-1), holons (-1), and DTs ($+2$) [6, 8]. The holon number is the dopant concentration, $\Sigma_i 2n_{hi}/N = 2n_h/N = x$, which specifies the "charge sector" of the system, another global constraint. Encoded as a gauge principle, the sole local constraint corresponds to a U(1) gauge theory [35], and hence in the pyrochlore QSL, even with doping, only one U(1) symmetry emerges from the local physics of the dimers and quasiparticles.

To make this gauge symmetry completely rigorous, and its connection to "strings" [23] more transparent, we express the system as a lattice gauge theory following Ref. [36]. The presence or absence of a dimer connecting any two sites i and j in the same tetrahedron is denoted by the states $\sigma_{ij}^z = \pm 1$, with corresponding singlet creation and destruction operators $\sigma_{ij}^\pm = \frac{1}{2}(\sigma_{ij}^x \pm i\sigma_{ij}^y)$. Any basis state may be specified only by σ^+ operators for all pairs of sites in the dimer covering a, $|\psi_a\rangle = \Pi_a \sigma_{\langle ij\rangle\in a}^+|0\rangle$. Any loop process between two such states is specified by a sequence $\sigma_{ab}^+\sigma_{bc}^-\sigma_{cd}^+\sigma_{de}^-\sigma_{ef}^+\sigma_{fg}^-\sigma_{gh}^+\sigma_{ha}^-$ connecting the dimers along alternate bonds, with two oper-

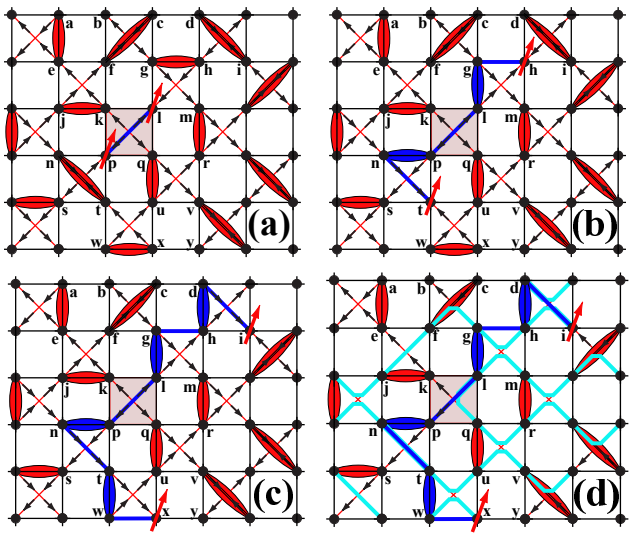

FIG. 4. (color online) Representation of (a) a single defect tetrahedron with two quasiparticles on the DT, (b) the first step for each quasiparticle away from the DT, and (c) the second, leaving a string. (d) Line representation and gauge string shown for the same quasiparticle propagation process.

ators in each tetrahedron. A quasiparticle is specified by $...\sigma^-_{ab}\sigma^+_{bc}\sigma^-_{cd}\overline{q}_d\sigma^-_{de}\sigma^+_{ef}\sigma^-_{fg}...$, where the two σ^- operators appearing in sequence encode its presence on site d ($\overline{q}_d$ is required to specify a spinon or holon).

Creating a DT in the basis wave function $|\psi_a\rangle$ replaces σ^+_{lp} by $\overline{q}_l\sigma^-_{lp}\overline{q}_p$ in Fig. 4(a). The local processes causing the two quasiparticles to move away from the DT are $\overline{q}_h\sigma^-_{gh}\sigma^+_{gl}\sigma^-_{lp}\sigma^+_{pn}\sigma^-_{nt}\overline{q}_t$ [Fig. 4(b)] and then $\overline{q}_i\sigma^-_{di}\sigma^+_{dh}\sigma^-_{gh}\sigma^+_{gl}\sigma^-_{lp}\sigma^+_{pn}\sigma^-_{nt}\sigma^+_{tw}\sigma^-_{wx}\overline{q}_x$ [Fig. 4(c)]. Schematically, the propagation of the quasiparticles develops an extended object, a string. Mathematically, this exact spin representation is transformed to a lattice gauge theory by $\sigma^\pm = e^{\pm iA_{ij}}/\sqrt{2}$ with a real, compact phase A_{ij}, which is canonically conjugate to σ^z_{ij}, as the lattice gauge field [36]. Destroying the dimer on a tetrahedron is described by $A_{ij} \longrightarrow -A_{ij} + \theta_i + \theta_j$, where θ_i is a U(1) phase degree of freedom carried by the quasiparticle $\overline{q}_i$ created on site i. As the quasiparticles propagate, lengthening their string (Fig. 4), this U(1) phase information is retained by the moving particles and its memory, effectively the fingerprint of the missing dimer, is preserved in the phase $-A_{ij}$ on the DT pinning the string.

The lattice gauge theory of Ref. [36] is an exact representation of the ground manifold of dimer coverings. In the presence of quasiparticles, introduced as above, its extension is a matter-coupled gauge theory of the form $\mathcal{L} = \overline{q}_i U_{ij} q_j$ + c.c.. This is a "minimal-coupling" gauge theory where the fermionic quasiparticles appear as source terms of the gauge field and the Lagrangean is invariant for all lattice sites under the U(1) gauge symmetry $U_{ij} \to \eta^*_i U_{ij}\eta_j$, $q_i \to q_i\eta^*_i$, with $\eta_i \equiv e^{i\theta_i}$.

However, it is essential to note here that the strings, or gauge fields, carry no energy in the pyrochlore QSL. As a consequence of the massive degeneracies of the ground and first-excited manifolds, all quasiparticle locations are entirely equivalent in energy and so spinon and holon motion is deconfined. The role of strings is to mediate the phase relationship connecting the quasiparticles, via their point of origin at a DT, but this U(1) phase does not correspond to a physical observable.

VI. STRINGS AND PHYSICS

A. Strings and Lines

The strings (of σ operators) in Fig. 4 are a transparent way to consider quasiparticle propagation. There are two σ operators on every tetrahedron of the string except for the DT, where there is only one. These strings are not the same as the lines used in the line representation [8], as shown in Fig. 4(d). Strings encode the U(1) phase information of the spinon/holon pair and the DT, i.e. they are specific to defect formation and phase conservation laws. Lines include all dimers and encode their local conservation law. The maximally flippable submanifold is the set of all states with an average of 1 line per tetrahedron [Fig. 1(a)], which is the most degenerate sector.

However, spinons and holons are both the endpoints of both lines and strings (Fig. 4). There is only one type ("flavor") of line and one type of string, which may end with either type of quasiparticle, and local processes allow quasiparticles to exchange the lines/strings of which they are endpoints. Both lines and strings are pinned to DTs. Lines do not cross, whereas strings from different pairs of quasiparticles may do so. Both lines and strings are useful graphical representations of the fact that both species of quasiparticle, although energetically deconfined (free to move anywhere at zero energy cost), do retain a memory of their origin. As one consequence of this, the two quasiparticles from the same DT cannot repair each other's tracks, whereas quasiparticles of different origins may do so in part.

B. Strings and Local Quasiparticles

In Secs. III and IV, we treated the quasiparticles as local objects in order to deduce their statistics, and in fact related them directly to local electron operators. However, in Sec. V we discussed their connection to strings, which suggests the relevance of extended objects. Indeed, it was pointed out in Ref. [23] that emergent fermions are non-local objects and should always be considered in pairs, whence our consideration of doped holon pairs in Sec. IV.

There are two issues to discuss here. First, on the question of whether quasiparticles are extended objects or not, the answer is already clear from Sec. V. Quasiparticles are deconfined objects and their gauge strings have no energy, acting only as a book-keeping device for

their phase. This defect-related phase variable is not a physical observable and does not appear in the quasiparticle statistics. DTs also have no dynamics and make no contributions to the statistics. Thus the quasiparticles can be treated as local objects. The only conceptual point here is the rather trivial one that it does not make sense to attempt to define the statistics using only one quasiparticle, and that this must be done by considering pairs.

The second issue is how to dope holes into a dimer-based system. Clearly, quasiparticles always replace dimers and thus by definition are introduced in pairs. Spinons are created automatically in pairs by the excitation of a single dimer; this process also creates one DT, with which the spinons maintain a fixed phase relation, but none of their physical properties are affected by it. By contrast, in the case of holons it is necessary to define a valid creation process, because in principle a single holon can be inserted by the elimination of a single electron from the ground manifold. However, this cannot be the complete process in a dimer-based system, such as the pyrochlore QSL, because the other electron from the destroyed dimer remains present; from Sec. IV, it will be a fully spin-polarized electron, i.e. a spinon. To avoid spinon interference, holon statistics are best considered by introducing them in pairs that replace a single dimer, also creating one DT, as in Sec. IV.

When a single hole is introduced in the system, its lonely ex-partner spinon may form a new singlet, the new unpaired spin forms another new singlet, and so on, in a sequence of reconstructed spin correlations equivalent to the propagation of the spinon away from the doped holon. Despite the appearance of an extended object, as above this string has no physical meaning. For the purposes of deducing exchange statistics, the quasiparticles in the exchange process originate in general from different DTs; whether the holons of an exchanged pair were introduced singly or pair-wise is not relevant, as long as single-hole doping is taken to replace each bosonic dimer by one holon and one spinon [6]. This satisfies the constraints that two quasiparticles are always accompanied by one DT and that the total number of quasiparticles is never odd. The possibility that the total number of spinons, or of holons, is individually odd has no effect on exchange considerations in the assumed limit of dilute quasiparticles. We reiterate that the process of creating quasiparticles is a fractionalization of dimers and our contribution here is to prove that both fractions are fermionic.

VII. SUMMARY

We have considered the nature of quasiparticles in the pyrochlore quantum spin liquid. Both spinons and holons have fermionic statistics. These properties are conferred entirely by the dimers of the highly degenerate basis manifold and are completely universal for all electronic dimer states. Spinons and holons are linked by strings, which correspond to a gauge field whose origin lies in the conservation of dimer number.

The lattice gauge theory of the pyrochlore QSL is not only an effective description but an exact recasting of the fact that all dimer states and loop processes are known exactly. The local ($d = 0$) U(1) gauge field expresses the local constraint and is connected with the emergent pyrochlore photon [35], a mode whose gaplessness is a consequence of the degeneracy of the ground manifold and whose linearly dispersive character is determined by the nature of the excited manifolds. Off the Klein point, additional planar ($d = 2$) gauge terms specify the new ground manifold [6]. In the presence of spinons or holons, one has a matter-coupled gauge theory with minimal coupling ($\overline{q}_i U_{ij} q_j$ + c.c.) of fermionic quasiparticle matter to a dimer-mediated bosonic field and U(1) gauge symmetry.

Concerning possible superconductivity in the doped pyrochlore QSL, minimal renormalization of the hopping t means significant kinetic energy may be gained by holon motion. A small concentration of holes doped into the insulating half-filled system will create a small Fermi surface. Given the attractive but weak potential caused by the immobile DTs [6], it appears likely that holons, being fermionic, will pair and then superconduct at suitably low temperatures, presenting a specific example of RVB-type superconductivity [13] at low doping.

We close with a brief discussion of the three key features exhibited by the pyrochlore QSL. The fundamental property of the pyrochlore geometry is its four-site unit, which allows each to contain one dimer in the ground manifold. This dimer-based structure establishes the local constraint determining the quasiparticle statistics. The constraint also links the creation of propagating quasiparticles to the two sites of their DT. Schematically, the composite operation $e^{i\theta_i}\overline{q}_i e^{i\theta_j}\overline{q}_j e^{-i\theta_{ij}} d_{ij}$ has a phase degree of freedom restricted by $\theta_i + \theta_j - \theta_{ij} = 0$, where θ_i is a U(1) quasiparticle phase and $\theta_{ij} = 2A_{ij}$ is given by the lattice gauge field. This link remains present as a gauge string and the memory of the missing dimer (violation of the constraint) is preserved in the wave function through A_{ij}. Finally, the fermionic nature of both spinons and holons is no big mystery. Both are the physical quasiparticles of the starting Hubbard model. There is no sense in which one is fractionalizing an electron, only pairs of electrons (dimers). The resulting fractions are fermions, the spinon having the essential characteristics of the electron and the holon of a missing electron.

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